be less by factor of $0.7h/R$. The new line

$$[Eh^3/24(1-\nu^2)R^4][-2(1-\nu)u'(w^{\cdot\cdot}-v^{\cdot}) + 2(v^{\cdot}+w)(w''+w^{\cdot\cdot}-v^{\cdot}) - \tfrac{3}{8}(1-\nu)(u^{\cdot}+v')^2 + (1-\nu)(u^{\cdot}+v')(w'^{\cdot}+\tfrac{1}{4}u^{\cdot}+\tfrac{3}{4}v') + (v^{\cdot}+w)^2]$$

when combined with the second original line yields the complete modified functional

$$P_2[\mathbf{u}] = \int \{[Eh/2(1-\nu^2)R^2][u'^2 + (v^{\cdot}+w)^2 + 2\nu u'(v^{\cdot}+w) + \tfrac{1}{2}(1-\nu)(u^{\cdot}+v')^2] + [Eh^3/24(1-\nu^2)R^4] \times [w''^2 + w^{\cdot\cdot 2} + 2\nu w''w^{\cdot\cdot} + 2(1-\nu)w'^{\cdot 2} + 2w(w''+w^{\cdot\cdot}) + w^2 + 2(1-\nu)(u'v^{\cdot} - u^{\cdot}v' + u^{\cdot}w'^{\cdot} - u'w^{\cdot\cdot} + v^{\cdot}w'' - v'w'^{\cdot})]\}dS \quad (3)$$

Conclusion

Although the aforementioned modifications produce no apparent simplification, Eq. (3) will, when load terms are included and variational methods applied, yield equilibrium equations which are relatively convenient to work with. Such equations, expressed in terms of the shell deflections and their derivatives can be combined, as is frequently done, into two fourth-order equations relating the tangential and longitudinal displacements independently to the radial displacements, and one eighth-order equation in terms of radial displacements only. The three equations thus obtained are in the form presented by Morley.

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Planar Motion of a Large Flexible Satellite

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Introduction

Utilizing the method developed in Refs. 8 and 9 for the stability analysis of coupled rigid-elastic systems, the effect of flexible antennas upon the stability of motion of a gravity-gradient satellite is investigated. The satellite is assumed to consist of a compact rigid-body containing two antennas located at 180° from each other and in the plane of the orbit. The conditions for stability are found to include the well-known rigid-body stability criteria, and in addition, requirements on the elastic and coupled rigid-elastic motion.

The importance of the elastic degrees of freedom was discovered with the unexpected tumbling of Explorer I in 1958. This led to the work of Thomson and Reiter[1] who were the first to show, in a somewhat heuretic fashion, that elastic energy could adversely affect the motion of a spacecraft. Subsequent analysis were carried out,[2–7] but in most cases, assumptions were made which restricted the results, and the analysis techniques used were not easily extended to cases other than the one being analyzed. Recently, however, two somewhat similar methods for the stability analysis of a system made up of rigid and elastic elements were developed simultaneously by Meirovitch[8] and Budynas.[9] Both methods are based on the Lyapunov direct method and provide a general and rigorous approach for the stability analysis of mechanical systems made up of rigid and elastic elements. The details of the two approaches differ somewhat in that 1) different reference coordinate systems are used, and 2) Ref. 8 makes use of eigenvalues rather than eigenfunctions to perform the stability investigation. Meirovitch's use of eigenvalues rather than eigenfunctions, however, does avoid one of the difficulties encountered in Ref. 9, namely, that of determining the proper truncation of the modal series However, for complicated systems, these eigenvalues will have to be obtained by approximate methods or experimentally, and thus any great advantage may be lost. As an illustration of the method developed, Meirovitch[8] examines the case of a gravity-gradient stabilized satellite with spin, while Budynas[9] first studies the case of planar motion and then investigates the case of three-dimensional motion without spin.

Since a three-dimensional application of the technique has already been published[8] this paper summarizes the investigation of the planar case carried out in Refs. 9 and 10. The results of the planar case are of interest since they provide an uncomplicated comparison between the well known stability criteria of a rigid-body gravity-gradient stabilized satellite and the additional criteria imposed by considering the elastic degrees of freedom.

Analysis

Because the method for stability discussed by Meirovitch[8] is similar to the one used here, this section will only briefly recapitulate the stability method to be used. Reference 8 or 10 can be consulted for a more detailed discussion. The Hamiltonian for a coupled rigid-elastic system is given by

$$H = \sum_i \dot{q}_i \partial L_1/\partial \dot{q}_i + \int \Big(\sum_i \dot{q}_i \partial \pounds_1/\partial \dot{q}_i + \sum_j \dot{\eta}_j \partial \pounds_1/\partial \dot{\eta}_j \Big) \times dx - L \quad (1)$$

where q_i and $\dot{q}_i$ are the rigid-body's generalized position and velocity; η_j and $\dot{\eta}_j$ are the elastic displacement and the rate of change of elastic displacement, and

$$L = T - V = L_1(q_i,\dot{q}_i) + \int \pounds_1(q_i,\dot{q}_i,\eta_j,\dot{\eta}_j,x,\partial\eta_i/\partial x,\ldots)dx \quad (2)$$

In Eq. (2), L_1 is that part of the Lagrangian that can be expressed as a function of the rigid-body terms alone, and $\pounds_1$ is that part that can be expressed as a function of the coupled rigid-elastic terms.

If it is assumed that the elastic displacement of the system η is measured with respect to a rigid-body reference axes such that $\eta = \eta(x)$, then the position of the ith elastic elements, with respect to an inertial reference frame is expressible in the form

$$\mathbf{r} = \mathbf{r}_i[q_1,q_2,\ldots,q_n,x,\eta(x),t]$$

where $q_1,q_2,\ldots,q_n$ are the rigid-body generalized coordinates. In this case the kinetic energy of the system can be written

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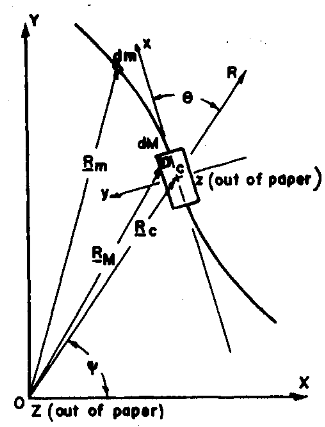

Fig. 1 Planar model.

in the form

$$T = T_2 + T_1 + T_0 = (T_{2R} + \mathfrak{T}_2) + (T_{1R} + \mathfrak{T}_1) + (T_{0R} + \mathfrak{T}_0) \quad (3)$$

where T_2 is quadratic and T_1 is linear in the velocities while T_0 is a function of the coordinates only. T_{0R}, T_{1R}, and T_{2R} refer to that portion of the kinetic energy that is a function of the rigid-body coordinates only, while $\mathfrak{T}_0$, $\mathfrak{T}_1$, and $\mathfrak{T}_2$ are functions of the spatial variable x and the coupled rigid-elastic and elastic terms.

In this case, the Hamiltonian can be expressed as

$$H = T_2 + V - T_0 = T_2 + U \quad (4)$$

where V is the potential energy of the system, expressible in the form

$$V = V_{\text{rigid}} + \int \mathfrak{V}_E dx \quad (5)$$

The term $\int \mathfrak{V}_E dx$ is the elastic potential energy of the system. The Dynamic Potential U can be expressed as

$$U = \int \mathfrak{U} dx + U_{\text{rigid}} \quad (6)$$

The equilibrium positions of the system can be found by investigating the steady-state portion of the equations of motion, which are[10]

$$\partial \mathfrak{U}/\partial \eta_j - \partial/\partial x[\partial \mathfrak{U}/\partial \eta_j'] + \partial^2/\partial x^2[\partial \mathfrak{U}/\partial \eta_j''] = \mathfrak{Q}_{xj} \quad (7a)$$

$$\partial U/\partial q_i = Q_i \quad (7b)$$

where primes denote differentiation with respect to x; $U = V - T_0$, and $\mathfrak{U} = \mathfrak{V}_E - \mathfrak{T}_0$ denote the dynamic potentials of the total system and the elastic components, respectively.

If the Lyapunov function V_L is chosen as

$$V_L = H - H_0 \quad (8)$$

where H_0 is the Hamiltonian at equilibrium, then since T_2 is zero at equilibrium, and $U = U_0$

$$V_L = T_2 + U - U_0 \quad (9)$$

For the system being considered here, $Q_i = 0$ and $\mathfrak{Q}_{xj}$ is taken as the force due to structural damping. In this case it is possible to show that $\dot{V}_L < 0$ and that the coupled rigid-elastic motion exhibits pervasive damping[10]; that is, $V_L = 0$ for all time implies the system is in equilibrium. For this situation the following theorem holds,[11,12] namely, Theorem: The equilibrium solution of a mechanical system with pervasive damping is 1) asymptotically stable, if V_L is a positive definite function, or 2) unstable, if V_L can take on negative values arbitrarily close to the equilibrium.

Since T_2 is known to be positive definite, then in order for V_L to be positive definite $U - U_0$ must be positive definite. For a rigid-body, testing $U - U_0$ for positive definiteness reduces to the problem of showing that the following determinants are greater than zero; that is,

$$D_1 = a_{11} > 0$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$

$$D_J = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{j1} & & & a_{jj} \end{vmatrix} > 0 \quad (10)$$

where $a_{ij} = \partial^2 U/\partial q_i \partial q_j$ is evaluated at equilibrium.

If the system has elastic degrees of freedom, the continuous coordinate $\eta_j(x,t)$ represents the deformations of an infinite collection of particles. Hence, the matrix a_{ij} is infinite and the determination of positive definiteness as described previously is not possible; unless the system is made discrete in some manner. To accomplish this we consider the elastic motion to be a linear combination of comparison functions χ_{lj} and time dependent generalized coordinates $\tau_{lj}(t)$. Thus,

$$\phi_j(x_k,t) = \sum_l \chi_{lj}(x_k)\tau_{lj}(t) \quad (11)$$

Truncation of the previous expression will yield a finite number of generalized coordinates.

Stability Analysis with Linear Elastic Appendages

Considering the Earth's center to be fixed, a point on the satellite can be referenced with respect to the Earth using the coordinates R, and ψ (Fig. 1). The satellite's reference system, xyz, is constructed such that when the appendages are in their undeformed state, xyz is a principal axis system whose origin is located at the center of mass of the composite satellite.‡ The orientation of the satellite in space is established by the angle θ, which relates the xyz and $R\psi z$ coordinate systems. Motion will be considered to take place in the xy or $R\psi$ plane only.

The kinetic energy T for the system shown in Fig. 1 is given by the expression[10]

$$T = \tfrac{1}{2}(M + 2m)(\dot{R}_c^2 + R_c^2\dot{\psi}^2) + \tfrac{1}{2}C'(\dot{\psi} + \dot{\theta})^2 + \tfrac{1}{2}\int_L\{\dot{u}^2 + \dot{v}^2 + (2ux + u^2 + v^2)(\dot{\psi} + \dot{\theta})^2 + 2\dot{R}_c\langle[\dot{u} - v(\dot{\psi} + \dot{\theta})]\cos\theta - [\dot{v} + u(\dot{\psi} + \dot{\theta})]\sin\theta\rangle + 2R_c\dot{\psi}\langle[\dot{u} - v(\dot{\psi} + \dot{\theta})]\sin\theta + [\dot{v} + u(\dot{\psi} + \dot{\theta})]\cos\theta\rangle + 2(\dot{\psi} + \dot{\theta})[\dot{v}(x + u) - u\dot{v}]\}dx \quad (12)$$

where M is mass of the main satellite, m the mass of one antenna, u and v the elastic displacements measured with respect to the rigid-body axis system, C' the moment of inertia of the complete undeformed satellite about the z axis, and R_c, ψ, and θ are as shown in Fig. 1. Integration within the limits $-l_0$ to $-\ell$ and l_0 to ℓ is implied by the notation $\int_L$.

The total potential energy consists of two parts; gravity and elastic potential

$$V = V_G + V_E = V_G + \int \mathfrak{V}_E dx \quad (13)$$

The gravity potential can be written as the sum of the gravity potentials of the main body and of the appendages, and is

‡ The xyz axis system has its origin at the center of mass of the composite satellite and its angular motion is defined to be that of the rigid-body motion of the satellite. The deflections of the elastic antenna are measured from this rigid-body axis system. Implicitly assumed here is that the mass of the main body is much greater than the mass of the appendages and that the motion of the center of mass of the system is unaffected by the attitude motion of the spacecraft.

given by the expression[10]

$$V_G = -K/R_c(M + 2m) + \tfrac{1}{4}K/R_c^3(A + B' + C') - 3K/4R_c^3[(C' + B' - A)\cos^2\theta + (C' + A - B')\sin^2\theta] + \tfrac{1}{2}K/R_c^3\int_L \times [(2ux + u^2)(1 - 3\cos^2\theta) + v^2(1 - 3\sin^2\theta) + 3v(x + u)\sin 2\theta + 2R_c(u\cos\theta - v\sin\theta)]dm \quad (14)$$

where A, B, and C are the moments of inertia of the rigid-body about the x, y, and z axes, respectively, and B',C' are the moments of inertia of the complete undeformed satellite about the y and z axes, respectively.

The elastic potential V_E for an inextensible beam is due to bending alone, and can be written as

$$V_E = \tfrac{1}{2}\int_L EI(\partial^2 v/\partial x^2)^2 dx \quad (15)$$

The Lagrangian $L = T - V$ is obtained by combining Eqs. (12, 14, and 15).

At this point it is assumed that the orbit of the center of mass of the undeformed vehicle is circular and that the beam antennas are uniform so that $dm = m/(l - l_0)dx$. Thus, substituting the Lagrangian into Eq. (1), the Hamiltonian assumes the form (4) and the dynamic potential U can be shown to be[10]

$$U \doteq -\tfrac{3}{4}\dot{\psi}_0^2(B' - A)\cos 2\theta + \tfrac{1}{2}\int_L\{3m/(\ell - \ell_0)\dot{\psi}_0^2[\tfrac{1}{2}(\ell^2 - x^2)(\partial v/\partial x)^2\cos^2\theta - v^2\sin^2\theta + vx\sin 2\theta] + EI(\partial^2 v/\partial x^2)^2\}dx \quad (16)$$

In Eq. (16) constant terms have been ignored and the antennas were assumed to be inextensionable.

The equilibrium conditions are found by substituting U into Eq. (7). Thus,

$$(B' - A)\sin 2\theta - m/(l - l_0)\{\sin 2\theta \int_L[\tfrac{1}{2}(l^2 - x^2)(\partial v/\partial x)^2 + v^2]dx - 2\cos 2\theta \int_L xv\, dx\} = 0 \quad (17)$$

and

$$EI\,\partial^4 v/\partial x^4 - \tfrac{3}{2}m/(l - l_0)\dot{\psi}_0^2\{\cos^2\theta\,\partial/\partial x[l^2 - x^2)\partial v/\partial x] + 2v\sin^2\theta - x\sin 2\theta\} = 0 \quad (18)$$

with boundary conditions $v = \partial v/\partial x = 0$ at $x = \pm l_0$; $\partial^2 v/\partial x^2 = \partial^3 v/\partial x^3 = 0$ at $x = \pm l$.

If the satellite is considered to be rigid, Eqs. (17) and (18) yield equilibrium points at $\theta_0 = 0°$, $90°$, $180°$, and $270°$, identified as E_1, E_2, E_3, and E_4, respectively (assuming $B' - A \neq 0$). The stability of a rigid-body vehicle in these equilibrium positions is, of course, well-known. With the appendages considered elastic, the additional equilibrium condition $v_0 = 0$ is required. Thus, the equilibrium conditions considered are

E_1,E_3	$\theta_0 = 0°$ or $180°$	$v_0(x) = 0$
E_2,E_4	$\theta_0 = 90°$ or $270°$	$v_0(x) = 0$

and an investigation of these positions will give a direct comparison between a rigid and a rigid-elastic vehicle in a configuration corresponding to gravity-gradient stabilization.

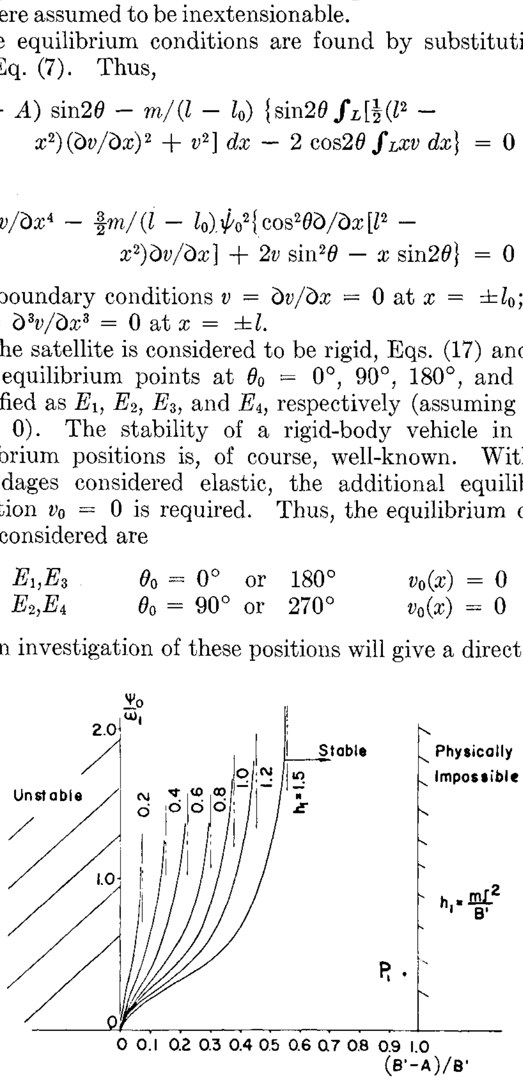

Fig. 2 Stability plot for E_1 and E_2.

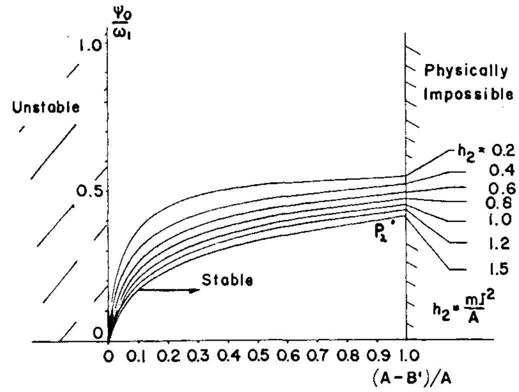

Fig. 3 Stability plot for E_2 and E_4.

In order to form a finite stability matrix, the elastic motion $v(x,t)$ is considered to be a linear combination of comparison functions $\chi_k(x)$ and time-dependent generalized coordinates $\tau_k(t)$. Thus,

$$v(x,t) = \sum_{k=1}^{\infty}\chi_k(x)\tau_k(t) \quad (19)$$

The comparison functions in this case are chosen as the eigenfunctions of the classical fixed-free beam, which are

$$\chi_k(x) = 0 \quad -l_0 < x < l_0$$
$$\chi_k(x) = \cosh\beta_k(x - l_0) - \cos\beta_k(x - l_0) - \gamma_k[\sinh\beta_k(x - l_0) - \sin\beta_k(x - l_0)]\, x \geq l_0 \quad (20)$$
$$x \leq -l_0$$

For practical reasons, the stability matrix is reduced to a finite matrix by truncating the series (19). If only one mode is retained for beams 1 and 2 and the orthogonality properties of Eq. (20) are utilized, the stability matrices for the equilibrium points under investigation can be found, and Sylvester's criterion applied.

In this case the stability requirements for E_1 and E_3 become

a) $(B' - A)/B' > 0$

b) $(\dot{\psi}_0/\omega_1)^2 < \{[3(B' - A)/B]/[5.951\, ml^2/B' - 15.666(B' - A)/B']\}$ (21)

where $\omega_1^2 = 12.362\, EI/ml^3$ is the fundamental natural frequency of a fixed-free beam, and the stability requirements for the E_2,E_4 positions become

a) $(A - B')/A > 0$

b) $(\dot{\psi}_0/\omega_1)^2 < \left\{\dfrac{[3(A - B')/A]}{[5.951\, ml^2/B' + 9.000(A - B')/A]}\right\}$ (22)

Requirement (a) in Eqs. (21) and (22) insures the stability of the rigid-body motion and agrees with the well known stability criterion for rigid-body satellites; whereas, requirement (b) insures the stability of the coupled rigid-body and elastic motion. Since there is positive damping, then if any of the stability conditions are not satisfied, the equilibrium point in question is unstable.

Figures 2 and 3 illustrate the regions of asymptotic stability and instability of the E_1,E_3 and E_2,E_4 equilibrium positions, respectively, in terms of the dimensionless orbital spin rate $\dot{\psi}_0/\omega_1$ (where ω_1 is the first natural frequency of the fixed-free beam) and a dimensionless inertia parameter. For a satellite, such as the RAE satellite, i.e., in an identical orbit and with similar antennas, then $\dot{\psi}_0/\omega_1 = 0.40$. For a typical inertia ratio of 0.95 points P_1 and P_2 in Figs. 2 and 3 represent characteristic positions in the stability diagrams. It can be seen that, for $h_{1,2} \cong 1.5$, P_1 is well into the stability zone, whereas P_2 is marginal.

It should also be pointed out that in order to evaluate the effect of additional modes on the stability requirements, two modes of antisymmetric elastic motion were also considered. However, the inclusion of these additional modes did not appreciably alter the stability criteria. The effect of relaxing the constrained orbit assumption was also investigated. The E_1,E_3 stability criteria remained unaffected, while the E_2,E_4 requirements were only slightly modified.

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Transonic Nozzle Flow with Nonuniform Total Energy

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Introduction

THE analysis of the performance of a convergent-divergent nozzle usually relies on a characteristic solution of the supersonic flowfield. Therefore, the transonic flow within the throat region must be known with sufficient accuracy to permit the development of a supersonic start line from which the generation of the characteristic net can be started. Calculating the transonic flow region for a potential flow presents no difficulty since this problem has received much attention and the results, notably those of Sauer[1] and Hall,[2] are well known. More recent studies[3–5] have also treated the flow in the transonic region; however, these too have been restricted to potential flow. There are instances in which a flow may have a rotational component, for example in a rocket nozzle with nonuniform total energy across the flow. In such cases, it has been a common practice to use the potential results to determine the start line in the absence of a solution for a rotational flow.

The purpose of this note is to present a solution for transonic flow in a nozzle throat with a variable stagnation speed of sound. The procedure used to solve the equations is similar to that used by Hall. However, the governing equations are written with the stream function as an independent variable to conveniently accommodate the variation of stagnation conditions across the flow.

Analysis

The origin of the cylindrical coordinate system (r,x) for the transonic region in an axisymmetric convergent-divergent nozzle will be taken at the geometric throat. The throat wall contour can be generally expressed as being circular, parabolic or hyperbolic. It is assumed that the variation in total temperature is produced upstream of the throat section. Then, neglecting transport phenomena, the flow can be considered isentropic along each streamline. However, the stagnation temperature (or stagnation speed of sound) is taken to depend upon the stream function. In addition, a uniform composition and a constant total pressure is assumed to exist throughout the flowfield.

It is then convenient to use $\xi = x/L$ and $\eta = \psi/\psi_w$ as independent variables where L is a characteristic axial length which will be defined later and $\psi_w = \dot{m}/2\pi$ is the value of the stream function corresponding to the wall of the nozzle. The transformation of the governing equations from x,r coordinates to ξ,η coordinates is accomplished with

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{L}\frac{\partial}{\partial \xi} - \frac{2\pi r}{\dot{m}}\rho v\frac{\partial}{\partial \eta} \qquad (1)$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{2\pi r}{\dot{m}}\rho u\frac{\partial}{\partial \eta} \qquad (2)$$

With the above transformation formulas, the continuity and Euler equations for axially symmetric compressible flow become

$$\frac{\partial(\rho u)}{\partial \xi} + \frac{2\pi r}{\dot{m}}\rho^2 L\left(u\frac{\partial v}{\partial \eta} - v\frac{\partial \mu}{\partial \eta}\right) + \rho\frac{vL}{r} = 0 \qquad (3)$$

$$\rho u\partial u/\partial\xi = -(\partial p/\partial\xi) + (2\pi r/\dot{m})\rho vL\partial p/\partial\eta \qquad (4)$$

$$\partial v/\partial\xi = -(2\pi r/\dot{m})L\partial p/\partial\eta \qquad (5)$$

Eliminating $\partial p/\partial\eta$ between Eqs. (4) and (5) yields

$$\rho u\partial u/\partial\xi + \rho v\partial v/\partial\xi = -(\partial p/\partial\xi) \qquad (6)$$

which if integrated would lead to the compressible Bernoulli equation along a streamline. Since the flow is isentropic along a streamline, we can write

$$\partial\rho/\partial\xi = (1/a^2)\partial p/\partial\xi = -(\rho/a^2)(u\partial u/\partial\xi + v\partial v/\partial\xi) \qquad (7)$$

and using this equation to eliminate the density derivative, Eq. (3) becomes

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial u}{\partial \xi} - \frac{uv}{a^2}\frac{\partial v}{\partial \xi} + \frac{2\pi rL}{\dot{m}}\rho \times \left(u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta}\right) + \frac{vL}{r} = 0 \qquad (8)$$

Eqs. (5) and (8) are to be solved for u and v where in terms of the total velocity and flow direction angle

$$u = V\cos\theta \quad \text{and} \quad v = V\sin\theta \qquad (9)$$

The speed of sound is given by

$$a^2 = [(\gamma + 1)/2]a^{*2}[1 - (\gamma - 1)/(\gamma + 1)V^{*2}] \qquad (10)$$

and for isentropic flow along a streamline the pressure and density can be expressed in terms of the stagnation pressure,

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